Problem 1
If f' is continuous on $[0, \infty)$ and $\lim_{x \to \infty} f(x) = 0$, show that

$$\int_0^\infty f'(x)\ dx = -f(0).$$

Problem 2
State the following convergence tests.

1. p-test.
2. Basic comparison test.
3. Limit comparison test.
4. Absolute convergence test.

Problem 3
Determine whether the following integrals converge. Do not find their value.

1. $\displaystyle\int_0^\infty \frac{\sin(x)}{x^2+1}\,dx.$
2. $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx.$
3. $\displaystyle\int_0^\infty \frac{1}{x+420}\,dx.$
4. $\displaystyle\int_0^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx.$
5. $\displaystyle\int_0^\infty \frac{3x}{x^3+x+2}\,dx.$
6. $\displaystyle\int_0^\pi \frac{\sin^2(x)}{\sqrt{x}}\,dx.$
7. $\displaystyle\int_0^1 \frac{1}{x^2+x}\,dx.$
8. $\displaystyle\int_0^1 \frac{1}{x^2+\sqrt{x}+2}\,dx.$

Problem 4
Define $\Gamma : (0, \infty) \to \mathbb{R}$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\ dt.$$

1. Show that $\Gamma(1) = 1$.
2. Show that $\Gamma(x+1) = x\Gamma(x)$.
3. Conclude that $\Gamma(n) = (n-1)!$ for $n = 1, 2, 3, \ldots$.